

Work, Power, Energy

Work

If a force is applied to a body, which then moves, we say the force does **work**.

In 1D, if the force is *constant* with magnitude F , and the body moves a distance x , the work done is

$$\boxed{W = Fx}.$$

The unit of work is the *joule* (symbol J).

1 joule = 1 newton metre (Nm),

which is the work done by a force of 1 newton moving through a distance of 1 metre.

(Note: 1 newton = 1 kg ms^{-2} , so 1 joule = $1 \text{ kg m}^2\text{s}^{-2}$.)

N.B. The distance moved is actually the displacement $s = x - x_0$. For the present we shall assume the starting position $x_0 = 0$ so that $s = x$.

Aside If the force is not constant then the work is calculated by dividing the displacement into small sections of length δx at position x . The work done in this small section is the force at x , $F(x)$ times the length

$$\delta W = F(x)\delta x$$

Adding all the sections gives the total work

$$W = \Sigma F(x) \delta x .$$

where Σ denotes summation. In the limit that the sections become infinitesimally small then $\delta x \rightarrow dx$ and $\Sigma \rightarrow \int$ so that

$$W = \int F(x) dx .$$

End of aside – We shall only consider constant forces in 0J2 so this will not be needed.

Power

Power P is defined by

$$\boxed{P = Fv},$$

where F is the force on a body, and v is its velocity.

This definition applies even if the force and/or velocity are changing. If the force is *constant* then $W = Fx$ and

$$\frac{dW}{dt} = F \frac{dx}{dt} = Fv = P$$

so in this case the power is the ‘rate of doing work’.

(One of the reasons why power is important in mechanics is that, for example, a car engine working at a fixed rate - at a fixed r.p.m. - generates (approximately) a fixed power; the force the engine generates will however vary with the speed of the car. As a car goes up a steep hill at constant power, it will slow down. As the velocity decreases the force produced by the engine will increase, until it is sufficient to maintain a constant (lower) velocity.)

The unit of power is joules/sec; this also has the name ‘watt’ (symbol W).

$$\begin{aligned} 1 \text{ watt} &= 1 \text{ joule/sec} = 1 \text{ Js}^{-1} \\ &= 1 \text{ Nms}^{-1} \\ &= 1 \text{ kg m}^2\text{s}^{-3} \end{aligned}$$

(Note: 1 kW = 1000W.)

Example: A car of mass 800 kg moves at a constant speed of 36 km/hr up a 1 in 8 incline. The top of the hill is a horizontal distance of 128 m from the car’s initial position. Find

- (i) The work done in getting to the top of the hill.
- (ii) The power at which the car works.

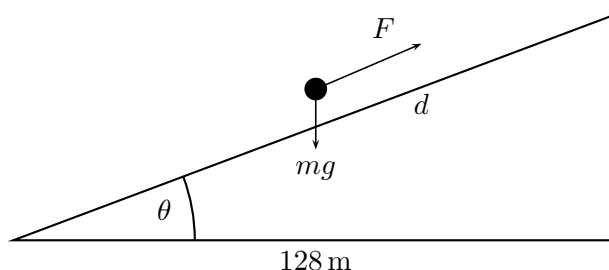
(Neglect friction, air resistance etc. Take $g = 9.81 \text{ ms}^{-2}$.)

Answer:

For a 1 in 8 slope:

$$\tan \theta = \frac{1}{8}, \quad \cos \theta = \frac{8}{\sqrt{65}},$$

$$\sin \theta = \frac{1}{\sqrt{65}}$$



Let the distance moved be d , so $d \cos \theta = 128$ and

$$d = \frac{128}{\cos \theta} = \frac{128\sqrt{65}}{8} = 16\sqrt{65}$$

Since speed is constant there is no net force (N1).

Consider forces parallel to plane:

$$F - mg \sin \theta = 0$$
$$F = mg \sin \theta = 800 \times g \times \frac{1}{\sqrt{65}}$$

So work done $W = \frac{Fd}{\sqrt{65}}$

$$= \frac{800g}{\sqrt{65}} \times 16\sqrt{65} = 12800 \times g = 125568\text{J}$$

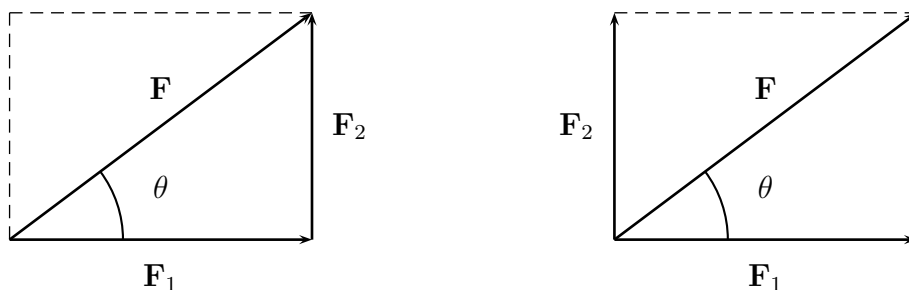
Power $P = Fv$

$$= \frac{800g}{\sqrt{65}} \times \frac{36 \times 1000}{3600}$$
$$= \frac{8000 \times 9.81}{\sqrt{65}} \approx 9734.2 \text{ watts}$$

Resolving Forces into components

Forces are vectors and can be split (resolved) into two perpendicular separate forces in more convenient directions.

Consider the vector sum of forces \mathbf{F}_1 and \mathbf{F}_2 using the triangle law. Let the sum be \mathbf{F} .



The second picture is equivalent to the first because vectors can be moved parallel to themselves without affecting their value.

Clearly the magnitudes of the three vectors satisfy

$$F_1 = F \cos \theta \quad \text{and} \quad F_2 = F \sin \theta$$

Note:

When we move through the angle θ we get a cosine

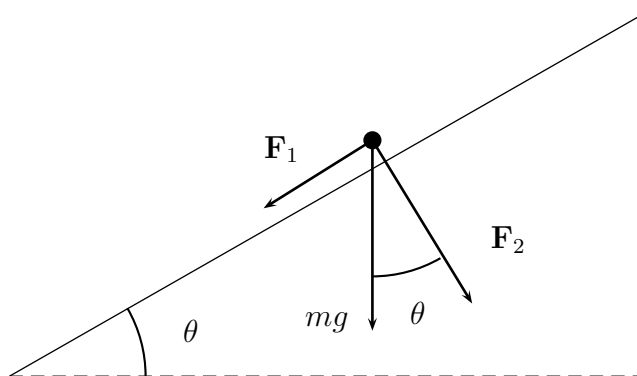
When we move away from the angle θ we get a sine.

Reversing this process we can now replace vector \mathbf{F} by the two vectors \mathbf{F}_1 and \mathbf{F}_2 arranged as in the second figure.

Now consider the weight (force of gravity) in the previous example.

This is a vertical force, with magnitude mg .

However we can split it into two forces parallel and perpendicular to the plane.



Here the weight (vector) has been replaced by vector \mathbf{F}_1 , parallel to the plane, and vector \mathbf{F}_2 , perpendicular to the plane.

Again note:

The magnitude F_2 of \mathbf{F}_2 is $mg \cos \theta$ since we move through the angle θ .

The magnitude F_1 of \mathbf{F}_1 is $mg \sin \theta$ since we move away from the angle θ .

Energy

If a body has the capacity (or ability) to do work we say it has **energy**. The energy of the body is the amount of work it can do.

When the body does some work it uses up some of its energy. But if work is done on the body its energy increases.

Energy comes in many forms (heat, light, electricity, ...) but we consider only *mechanical energy*. Mechanical energy is of two types:

- a) *kinetic energy* – ability to do work by virtue of having speed.
- b) *potential energy* – ability to do work by virtue of position.

Kinetic Energy

Suppose a particle of mass m is accelerated from rest to velocity v in a distance x by a constant force F . Here $u = 0$, so

$$v^2 = 0 + 2ax$$

But also $F = ma$ so

$$v^2 = 2\frac{F}{m}x$$
$$\therefore Fx = \frac{1}{2}mv^2$$

Force times distance is work done, so the work done in getting to speed v from speed 0 is $\frac{1}{2}mv^2$. This is called the **kinetic energy** of the particle, since if we now reverse the process the particle can do this amount of work in slowing down to rest.

$$\therefore \text{kinetic energy (KE)} = \frac{1}{2}mv^2 \tag{1}$$

Note: since work is measured in joules, so is energy.

Potential Energy

Potential energy is usually due to gravity. Suppose we lift a particle of mass m from height 0 to height h . (We do this infinitely slowly, so no KE is involved.) The force needed is mg , and the distance moved is h , so the work done is mgh .

Again, if we reverse the process the particle can do this amount of work in coming down (very slowly!).

$$\therefore \text{potential energy (PE) (due to gravity)} = mgh \tag{2}$$

Notes:

1. We only ever need changes in height, so it doesn't matter where we measure height from.
2. If h is $-ve$, then so is PE.

Another type of potential energy occurs when a spring is compressed, or an elastic string is stretched. Since we have to do work to do either of these, then reversing the process can cause work to be done.

Example 1: A particle of mass 2 kg is initially 3 m above a table of height 1 m. It is moved slowly down on to the table. Find the change in the PE.

$$\text{Initial height } h_1 = 4, \quad \text{so PE} = 2g \times 4 = 8g$$

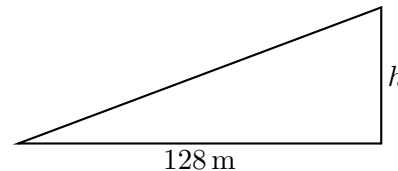
$$\text{Final height } h_2 = 1, \quad \text{so PE} = 2g \times 1 = 2g$$

so change in PE is $2g - 8g = -6g$ joules.

Note: Change in height is -3 m,
so change in PE is $mg \times (-3) = -6g$ as expected.

Example 2: Find the change in PE of the car in the earlier example, which moved up a hill.

Slope is 1 in 8; horizontal distance is 128 m



So vertical distance is h where

$$\frac{h}{128} = \frac{1}{8} \quad \therefore h = 16 \text{ m}$$

Change in PE is mgh . Taking $g = 9.81 \text{ ms}^{-1}$ we get

$$\text{Increase in PE} = 800 \times 9.81 \times 16 = 125568 \text{ joules}$$

Notes:

1. Increase in PE is exactly equal to work done (found earlier).
2. No change in KE since $v = 36 \text{ km/hr}$ at bottom and top.

Equivalence of Work and Energy

When work is done on a body by applying a force which moves through a distance, e.g. by

- raising it (or compressing spring, stretching elastic etc.)
- accelerating it (increasing its velocity)

then its energy increases by the amount of work done.

Similarly, a body which possesses energy, either kinetic or potential, can give up that energy by doing work. Hence we say

Work and Energy are equivalent

and they are both measured in joules.

Conservation of Energy

If no work is done on a body, then its energy is unchanged. We say:

If the total work done by external forces acting on a body is zero, there is no change in the total mechanical energy of the body.

This is called the principle of conservation of mechanical energy.

Notes:

1. The weight of a body does not count as an external force, since PE already takes this into account.
2. Although the total mechanical energy is unchanged, it can change from PE to KE or vice versa.

Example 1: A particle is dropped from rest at height h . With what velocity does it strike the ground?

Method 1: $u = 0$, $s = h$, $a = g$, so

$$\begin{aligned}v^2 &= u^2 + 2as \\ &= 0 + 2gh \\ \therefore v &= \sqrt{2gh}\end{aligned}$$

Method 2: At start $KE=0$, $PE=mgh$. At finish $KE=\frac{1}{2}mv^2$, $PE=0$.

By conservation of energy:

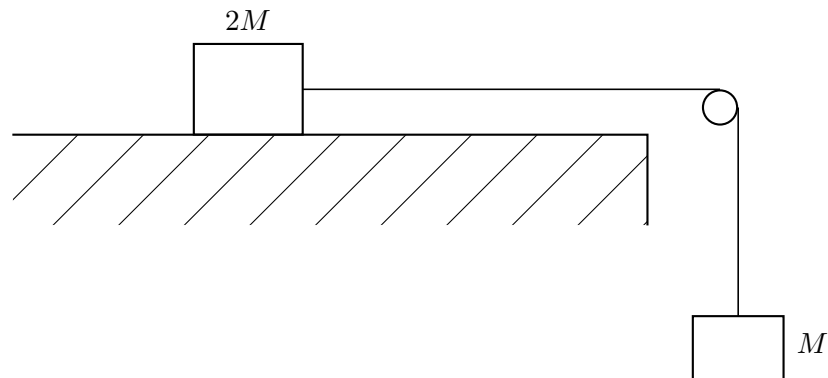
$$\begin{aligned}0 + mgh &= \frac{1}{2}mv^2 \\ \therefore v^2 &= 2gh \\ v &= \sqrt{2gh}\end{aligned}$$

Note: Measuring the height from a different point does not affect the answer, since conservation of energy says that

$$\underline{\text{change in KE}} = - \underline{\text{change in PE}}$$

so only the *change* in height matters.

Example 2: A block of mass $2M$ is lying on a smooth horizontal table. It is attached by a light, inelastic string, which passes over a pulley, to a mass M , as shown.



If the system is released from rest, find the velocity after M has dropped 0.8 m, with the $2M$ mass still on the table. (Take $g = 9.81 \text{ ms}^{-2}$).

Answer:

Let the final velocity be v . Both masses have this v , so final KE is

$$\frac{1}{2}Mv^2 + \frac{1}{2}(2M)v^2 = \frac{3}{2}Mv^2$$

Initial KE is 0, so change in KE is $\frac{3}{2}Mv^2$.

Change of PE of $2M$ is zero, since it is still on the table (has not changed height).

Change of PE of M is

$$Mgh = M \times 9.81 \times (-0.8) = -7.848M$$

Total change in energy is zero so

$$\begin{aligned} \frac{3}{2}Mv^2 - 7.848M &= 0 \\ \therefore v^2 &= \frac{2 \times 7.848}{3} = 5.232 \\ \therefore v &= \sqrt{5.232} = 2.287 \text{ ms}^{-1} \end{aligned}$$

Note we only needed the change in PE.

(This example can also be done using forces, tensions, accelerations etc., but the answer is longer.)